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一类七次多项式系统高次奇点的极限环分支与拟等时中心*

吴玉森， 李培峦

(河南科技大学数学与统计学院，洛阳 471003)

摘 要: 本文研究了一类七次多项式系统高次奇点的中心、拟等时中心条件与极限环分支问题．首先通过同胚变换和复变换将系统的高次奇点化为复域中的初等原点，然后求出了新系统在原点的前45个奇点量，从而导出了高次奇点为中心和最高阶细焦点的条件．在此基础上给出了七次系统在高次奇点分支出8个极限环的实例．最后通过一种新的算法求出高次奇点为中心时的周期常数，得到了高次奇点为拟等时中心的必要条件，并一一证明了这些条件的充分性．

关键词: 高次奇点；极限环分支；拟等时中心；七次系统

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1 引言

在平面多项式微分系统定性理论中，中心与等时中心这两个经典问题一直吸引着众多数学工作者的兴趣．关于初等奇点的中心与等时中心问题已有大量的研究结果[1-5]，但关于多项式系统高次奇点这方面的研究结果还很少见，目前仅有一些关于高次奇点稳定性、极限环分支的结果[6]．

研究系统原点中心条件的一种方法是先求出原点的前面若干个焦点量，然后置这些焦点量为零，从而获得原点为中心的必要条件，最后通过各种方法来证明这些条件的充分性．如果一个平面多项式系统的中心的充分小临域内闭轨族的周期为相同的常数，则称之为等时中心．类似地，当原点为中心时，研究系统等时中心的一种方法是先求出原点的前面若干个周期常数，然后置这些周期常数为零，从而获得原点为等时中心的必要条件．最后通过多种途径证明条件的充分性．计算焦点量的经典方法有Poincaré后继函数法和Lyapunov形式级数法[7]，而计算周期常数的方法有等时常数法[3,4,8]，周期常数法[9]，复系统的周期常数法[5]．以上计算焦点量和周期常数的方法都具有计算复杂、不便应用的缺点．刘一戎、李继彬[10]给出了一套统一计算焦点量和鞍点量为奇点量来解决复中心问题的新方法．在此基础上，刘一戎、黄文韬[11]又给出了一种在复域中计算复系统中心周期常数的新算法，即解决了寻找复等时中心必要条件的问题．上述计算奇点量和周期常数的新方法容易通过计算机代数软件Mathematica或Maple实现，只需以系统右端系数作为符号进行有限次加、减、乘、除的四则运算和符号推导，避免了复杂的非线性积分运算和求解多元方程组．

收稿日期: 2009-03-04. **作者简介**: 吴玉森 (1982年7月生)，男，博士，讲师．研究方向：常微分方程定性理论与计算机符号计算．

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本文研究如下七次多项式系统

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = (\delta x - y)(x^2+y^2) + X_5(x,y) - \lambda y(x^2+y^2)^3, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = (x + \delta y)(x^2+y^2) + Y_5(x,y) + \lambda x(x^2+y^2)^3, \end{cases} \tag{1}$$

其中

$$X_5(x,y) = \sum_{k+j=5} C_{kj}x^k y^j, \quad Y_5(x,y) = \sum_{k+j=5} D_{kj}x^k y^j, \tag{2}$$

$$\begin{aligned} &C_{50} = -B_{23} - B_{32} - B_{41} - B_{50}, && C_{41} = A_{23} - A_{32} - 3A_{41} - 5A_{50}, \\ &C_{32} = -2(B_{23} + B_{32} - B_{41} - 5B_{50}), && C_{23} = 2(A_{23} - A_{32} - A_{41} + 5A_{50}), \\ &C_{14} = -B_{23} - B_{32} + 3B_{41} - 5B_{50}, && C_{05} = A_{23} - A_{32} + A_{41} - A_{50}, \\ &D_{50} = A_{23} + A_{32} + A_{41} + A_{50}, && D_{41} = B_{23} - B_{32} - 3B_{41} - 5B_{50}, \\ &D_{32} = 2(A_{23} + A_{32} - A_{41} - 5A_{50}), && D_{23} = 2(B_{23} - B_{32} - B_{41} + 5B_{50}), \\ &D_{14} = A_{23} + A_{32} - 3A_{41} + 5A_{50}, && D_{05} = B_{23} - B_{32} + B_{41} - B_{50}. \end{aligned} \tag{3}$$

系统 (1) 的原点为高次奇点，且它的 Poincaré 闭球面的赤道 Γ_∞ 为系统的轨线，其上没有实奇点．本文通过同胚变换和复变换把系统 (1) 的高次奇点化为初等奇点在复域中来研究，得到了系统的中心条件和最高阶细焦点 (细奇点) 条件，构造出了一个在高次奇点分支出 8 个极限环的七次多项式系统实例．然后，在中心条件的基础上计算了系统的周期常数，得到了此系统高次奇点为拟等时中心的必要条件，最后通过多种有效途径证明了这些条件的充分性．这实际上解决了这类七次系统的伴随系统高次奇点的拟等时中心问题与其自身为实系统时鞍点的可线性化问题．

2 高次奇点的奇点量与中心条件

通过同胚变换

$$x = \xi(\xi^2+\eta^2)^2, \quad y = \eta(\xi^2+\eta^2)^2, \quad \mathrm{d}t = (\xi^2+\eta^2)^{-5}\mathrm{d}\tau, \tag{4}$$

系统 (1) 变为

$$\begin{cases} \frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \frac{\delta}{5}\xi - \eta + \Big[(\frac{\xi^2}{5}+\eta^2)(C_{50}\xi^5 + C_{41}\xi^4\eta + C_{32}\xi^3\eta^2 + C_{23}\xi^2\eta^3 \\ \qquad + C_{14}\xi\eta^4 + C_{05}\eta^5) - \frac{4}{5}\xi\eta(D_{50}\xi^5 + D_{41}\xi^4\eta + D_{32}\xi^3\eta^2 \\ \qquad + D_{23}\xi^2\eta^3 + D_{14}\xi\eta^4 + D_{05}\eta^5)\Big](\xi^2+\eta^2)^2 - \lambda\eta(\xi^2+\eta^2)^{10}, \\ \frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \xi + \frac{\delta}{5}\eta + \Big[(\xi^2+\frac{\eta^2}{5})(D_{50}\xi^5 + D_{41}\xi^4\eta + D_{32}\xi^3\eta^2 + D_{23}\xi^2\eta^3 \\ \qquad + D_{14}\xi\eta^4 + D_{05}\eta^5) - \frac{4}{5}\xi\eta(C_{50}\xi^5 + C_{41}\xi^4\eta + C_{32}\xi^3\eta^2 \\ \qquad + C_{23}\xi^2\eta^3 + C_{14}\xi\eta^4 + C_{05}\eta^5)\Big](\xi^2+\eta^2)^2 + \lambda\xi(\xi^2+\eta^2)^{10}. \end{cases} \tag{5}$$

再通过复变换

$$z = \xi + \mathrm{i}\eta, \quad w = \xi - \mathrm{i}\eta, \quad T = \mathrm{i}\tau, \quad \mathrm{i} = \sqrt{-1}, \tag{6}$$

系统 (5) 变为它的伴随系统

$$\left\{\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}T} &= \left(1-\tfrac{\delta}{5}\mathrm{i}\right)z+\tfrac{3}{5}a_{50}z^8w^3+\tfrac{1}{5}(3a_{41}+2b_{23})z^7w^4+\tfrac{1}{5}(3a_{32}+2b_{32})z^6w^5\\&\quad+\tfrac{1}{5}(3a_{23}+2b_{41})z^5w^6+\tfrac{2}{5}b_{50}z^4w^7+\lambda z^{11}w^{10},\\\frac{\mathrm{d}w}{\mathrm{d}T} &= -\left(1+\tfrac{\delta}{5}\mathrm{i}\right)w-\tfrac{3}{5}b_{50}w^8z^3-\tfrac{1}{5}(3b_{41}+2a_{23})w^7z^4-\tfrac{1}{5}(3b_{32}+2a_{32})w^6z^5\\&\quad-\tfrac{1}{5}(3b_{23}+2a_{41})w^5z^6-\tfrac{2}{5}a_{50}w^4z^7-\lambda w^{11}z^{10}.\end{aligned}\right. \tag{7}$$

显然，系统 (7) 的右端系数共轭，即

$$\begin{aligned}&a_{50}=A_{50}+\mathrm{i}B_{50},\quad a_{41}=A_{41}+\mathrm{i}B_{41},\quad a_{32}=A_{32}+\mathrm{i}B_{32},\quad a_{23}=A_{23}+\mathrm{i}B_{23},\\&b_{50}=A_{50}-\mathrm{i}B_{50},\quad b_{41}=A_{41}-\mathrm{i}B_{41},\quad b_{32}=A_{32}-\mathrm{i}B_{32},\quad b_{23}=A_{23}-\mathrm{i}B_{23}.\end{aligned} \tag{8}$$

利用文献 [12] 定理 2.1 给出的计算原点奇点量的一般递推公式，通过计算机符号运算与化简可得以下定理.

定理 1 系统 $(7)_{\delta=0}$ 原点的前 45 个奇点量为

$$\mu_5=\frac{1}{5}(a_{32}-b_{32}),\quad \mu_{10}=\frac{1}{5}(-a_{23}a_{41}+b_{23}b_{41}),$$

$$\mu_{15}=\frac{1}{40}(9a_{23}^2a_{50}-a_{50}b_{41}^2+a_{41}^2b_{50}-9b_{23}^2b_{50}),$$

$$\mu_{20}=\frac{1}{60}(a_{32}+b_{32})(3a_{23}a_{50}b_{41}-a_{50}b_{41}^2+a_{41}^2b_{50}-3a_{41}b_{23}b_{50}),$$

$$\mu_{25}=\frac{1}{3240}(-3a_{23}a_{50}b_{41}+a_{50}b_{41}^2-a_{41}^2b_{50}+3a_{41}b_{23}b_{50})(-16a_{41}b_{41}+27a_{50}b_{50}-216\lambda),$$

$$\mu_{30}=0,$$

$$\begin{aligned}\mu_{35}=-\frac{1}{2332800}&(3a_{23}a_{50}b_{41}-a_{50}b_{41}^2+a_{41}^2b_{50}-3a_{41}b_{23}b_{50})\\&(6400a_{41}^2b_{41}^2-8928a_{41}a_{50}b_{41}b_{50}+405a_{50}^2b_{50}^2),\end{aligned}$$

$$\begin{aligned}\mu_{40}=-\frac{7}{349920}&(a_{50}b_{41}^2+a_{41}^2b_{50})(-32a_{41}b_{41}+45a_{50}b_{50})\\&(-3a_{23}a_{50}b_{41}+a_{50}b_{41}^2-a_{41}^2b_{50}+3a_{41}b_{23}b_{50}),\end{aligned}$$

$$\begin{aligned}\mu_{45}=-\frac{11}{68890500}&a_{41}^2b_{41}^2(164000a_{41}b_{41}-224181a_{50}b_{50})\\&(3a_{23}a_{50}b_{41}-a_{50}b_{41}^2+a_{41}^2b_{50}-3a_{41}b_{23}b_{50}),\end{aligned}$$

其中 $\mu_k=0$, $k\neq 5i$, $i\leq 9$, $i\in N$. 在上述 μ_k 的表达式中已置 $\mu_1=\mu_2=\cdots=\mu_{k-1}=0$, $k=2,3,\cdots,45$.

定理 2 系统 $(7)_{\delta=0}$ 原点的前 45 个奇点量全部为零的充要条件是下列三组条件之一成立:

(I) $a_{32}=b_{32}$, $a_{41}=b_{41}=0$, $a_{23}^2a_{50}=b_{23}^2b_{50}$;

(II) $a_{32}=b_{32}$, $a_{23}=\frac{1}{3}b_{41}$, $b_{23}=\frac{1}{3}a_{41}$, $a_{41}b_{41}\neq 0$;

(III) $a_{32}=b_{32}$, $a_{23}a_{41}=b_{23}b_{41}$, $a_{41}^2b_{50}=b_{41}^2a_{50}$, $a_{41}b_{41}\neq 0$.

根据文献 [10]，系统 $(7)_{\delta=0}$ 有 14 个基本 Lie 不变量：

$$\begin{aligned}&a_{32},\ b_{32},\ \lambda\,(\text{i.e.},\ a_{43},\ b_{43}),\ a_{50}b_{50},\ a_{41}b_{41},\ a_{23}b_{23},\ a_{41}a_{23},\ b_{41}b_{23},\\&a_{41}^2b_{50},\ a_{41}b_{23}b_{50},\ b_{23}^2b_{50},\ b_{41}^2a_{50},\ b_{41}a_{23}a_{50},\ a_{23}^2a_{50}.\end{aligned}\tag{9}$$

定理 2 给出了系统 $(7)_{\delta=0}$ 的原点为中心的必要条件. 在定理 2 中，如果条件 (I) 或 (III) 成立，则系统 $(7)_{\delta=0}$ 满足广义对称原理[10] 的条件；如果条件 (II) 成立，记 $a_{32}=b_{32}=r_{32}$，则系统 $(7)_{\delta=0}$ 有通积分

$$F(z,w)=\frac{z^{15}w^{15}}{4+12\lambda z^{10}w^{10}+3a_{50}z^7w^3+4a_{41}z^6w^4+6r_{32}z^5w^5+4b_{41}w^6z^4+3b_{50}w^7z^3}.$$

所以，我们有以下定理.

定理 3 系统 $(7)_{\delta=0}$ 的原点 (相应地，系统 $(1)_{\delta=0}$ 的高次奇点) 为中心的充要条件是定理 2 中的三组条件之一成立.

3 高次奇点的极限环分支实例

由定理 1 容易得到：

定理 4 系统 $(7)_{\delta=0}$ 的原点为 45 阶细奇点，即 $\mu_1=\mu_2=\cdots=\mu_{44}=0$，$\mu_{45}\neq0$ 的充要条件是

$$\begin{aligned}&a_{32}=b_{32}=0,\quad a_{23}=-\tfrac{1}{3}b_{41},\quad b_{23}=-\tfrac{1}{3}a_{41},\quad 16a_{41}b_{41}-27a_{50}b_{50}+216\lambda=0,\\&6400a_{41}^2b_{41}^2-8928a_{41}a_{50}b_{41}b_{50}+405a_{50}^2b_{50}^2=0,\\&a_{41}^2b_{50}+b_{41}^2a_{50}=0,\quad a_{41}^2b_{41}^2(a_{41}^2b_{50}-b_{41}^2a_{50})\neq0,\end{aligned}\tag{10}$$

且细奇点的最高阶数为 45 阶.

由定理 1 及定理 4，我们构造出扰动系统 (7) 在原点分支出 8 个极限环的实例如下：

定理 5 如果系统 (7) 的系数满足

$$\begin{aligned}&\delta=\tfrac{1}{2}c_1\varepsilon^{90},\quad a_{32}=\tfrac{45}{8}\sqrt{\tfrac{5}{31-2\sqrt{209}}}c_{41}\varepsilon^{50}-\tfrac{5}{2}\mathrm{i}c_{11}\varepsilon^{80},\\&a_{41}=1,\quad a_{23}=-\tfrac{1}{3}+\tfrac{5}{2}\sqrt{\tfrac{5}{31-2\sqrt{209}}}c_{31}\varepsilon^{60}+\tfrac{5}{2}\mathrm{i}c_{21}\varepsilon^{70},\\&a_{50}=\tfrac{4}{3}\mathrm{i}\sqrt{\tfrac{1}{5}(31-2\sqrt{209})}-\frac{32805\sqrt{\frac{5}{31-2\sqrt{209}}}}{112(-29+2\sqrt{209})}c_{81}\varepsilon^{10}+\frac{3645\mathrm{i}(-29+2\sqrt{209})^2}{64(-58102+4019\sqrt{209})}c_{71}\varepsilon^{20}\\&\qquad-\frac{645700815\mathrm{i}\left(31\sqrt{\frac{5}{31-2\sqrt{209}}}-\sqrt{5(31-2\sqrt{209})}\right)}{200704(-58102+4019\sqrt{209})}c_{81}^2\varepsilon^{20},\\&\lambda=-\tfrac{4}{135}(-44+3\sqrt{209})-\frac{1215\sqrt{5(31-2\sqrt{209})}(-2695939+186482\sqrt{209})}{64(-467023876+32304717\sqrt{209})}c_{71}\varepsilon^{20}\\&\qquad+\frac{45\sqrt{5(31-2\sqrt{209})}(-40320698+2789041\sqrt{209})^2(-467023876+32304717\sqrt{209})}{3344(-87063226196609677046 7338+602228928310386488286 71\sqrt{209})}c_{51}\varepsilon^{40}\\&\qquad-\tfrac{1}{k_0}\left(c_{71}^2+c_{71}c_{81}^2-\frac{k_1(-k_1-k_2+k_3)}{k4}c_{81}^4\right)\varepsilon^{40},\\&b_{50}=\overline{a_{50}},\quad b_{41}=\overline{a_{41}},\quad b_{32}=\overline{a_{32}},\quad b_{23}=\overline{a_{23}},\end{aligned}\tag{11}$$

其中

$$c_1=\frac{173421898261576036442872641577070547593619289908077973986711961600000000000}{121067072456402641}c_{91},$$

$$c_{11}=-\frac{1215160629746042602240705060735437866470921996064992206035539753675134074 88}{847469507194818487}c_{91},$$

$$c_{21}=\frac{3622089796833358438491747317158615177263151507858437828829975136185600 00}{2542408521584455461}c_{91},$$

$$c_{31}=-\frac{305008887091951672959669262286858621270342673971056072900183062289}{121067072456402641}c_{91},$$

$$c_{41}=\frac{2132409968064926197884474045564923992160132099791928427657500}{847469507194818487}c_{91},$$

$$c_{51}=-\frac{589266448471796931383988714572049616152139635907886009}{2542408521584455461}c_{91},$$

$$c_{71}=\frac{340937183110805436340833407123598 7027}{847469507194818487}c_{91},$$

$$c_{81}=-\frac{117843678250533067519015325 00}{2542408521584455461}c_{91},$$

$$c_{91}=-\frac{704\sqrt{\frac{1}{5}(31-2\sqrt{209})}(-720929+49818\sqrt{209})}{258339375},$$

$$k_0=\frac{16(-870632261966096770467338\sqrt{5}+602228928310386488286 71\sqrt{1045})}{225(145314473405987573928 24\sqrt{31-2\sqrt{209}}-100516123064003658 2033\sqrt{209(31-2\sqrt{209})})},$$

$$k_1=56571532675374775209634720489081956705845341642426381674 2577,$$

$$k_2=11870639377320314685081645804652993977855377889615892440320 00\sqrt{5(31-2\sqrt{209})},$$

$$k_3=82029091977066179468629702882836292337913888289260735744000\sqrt{1045(31-2\sqrt{209})},$$

$$k_4=135560523097079573267220083802515738110294523645974750867004336801398367 41487$$
$$287856862019828660725432320000000(-31+2\sqrt{209})(-720929+49818\sqrt{209})^2,$$

相应地，系统(1)的系数可由(3),(8),(11)式定出，则当$0<\varepsilon\ll 1$时，系统(5)在原点的邻域内经微扰将产生8个极限环，其位置在圆$\xi^2+\eta^2=k^2\varepsilon^2$，$k=1,2,3,4,5,7,8,9$附近．相应地，系统(1)在高次奇点的邻域内经微扰将产生8个极限环，其位置在圆$x^2+y^2=k^{10}\varepsilon^{10}$，$k=1,2,3,4,5,7,8,9$附近．

证明　由定理1及$\nu_1(2\pi)=\mathrm{e}^{\frac{2}{5}\pi\delta}$，$\nu_{2m+1}(2\pi)\sim \mathrm{i}\pi\mu_m$，经过细致的计算我们得到

$$\begin{aligned}
&\nu_1(2\pi,\delta)-1=c_1\pi\varepsilon^{90}+o(\varepsilon^{90}), && \nu_{11}(2\pi,\delta)=c_{11}\pi\varepsilon^{80}+o(\varepsilon^{80}),\\
&\nu_{21}(2\pi,\delta)=c_{21}\pi\varepsilon^{70}+o(\varepsilon^{70}), && \nu_{31}(2\pi,\delta)=c_{31}\pi\varepsilon^{60}+o(\varepsilon^{60}),\\
&\nu_{41}(2\pi,\delta)=c_{41}\pi\varepsilon^{50}+o(\varepsilon^{50}), && \nu_{51}(2\pi,\delta)=c_{51}\pi\varepsilon^{40}+o(\varepsilon^{40}),\\
&\nu_{61}(2\pi,\delta)=o(\varepsilon^{90}), && \nu_{71}(2\pi,\delta)=c_{71}\pi\varepsilon^{20}+o(\varepsilon^{20}),\\
&\nu_{81}(2\pi,\delta)=c_{81}\pi\varepsilon^{10}+o(\varepsilon^{10}), && \nu_{91}(2\pi,\delta)=c_{91}\pi+o(1).
\end{aligned}\tag{12}$$

由(12)式可得Poincaré后继函数为

$$d(\varepsilon h)=r(2\pi,\varepsilon h)-\varepsilon h=\pi\varepsilon^{91}h\big[g(h)+\varepsilon hG(h,\varepsilon)\big],\tag{13}$$

其中 $G(h,\varepsilon)$ 在原点解析

$$\begin{aligned}
g(h) &= c_{91}(h^{10}-1)(h^{10}-2^{10})(h^{10}-3^{10})(h^{10}-4^{10})(h^{10}-5^{10})(h^{10}-7^{10})\\
&\quad (h^{10}-8^{10})(h^{10}-9^{10})\Big(h^{10}+\frac{556167241916424665616982789}{254240852158445546 1}\Big)\\
&= \frac{1}{2542408521584455461}c_{91}(-9+h)(-8+h)(-7+h)(-5+h)(-4+h)(-3+h)(-2+h)\\
&\quad \times(-1+h)(1+h)(2+h)(3+h)(4+h)(5+h)(7+h)(8+h)(9+h)\\
&\quad \times(6561-729h+81h^2-9h^3+h^4)(4096-512h+64h^2-8h^3+h^4)\\
&\quad \times(2401-343h+49h^2-7h^3+h^4)(625-125h+25h^2-5h^3+h^4)\\
&\quad \times(256-64h+16h^2-4h^3+h^4)(81-27h+9h^2-3h^3+h^4)(16-8h+4h^2-2h^3+h^4)\\
&\quad \times(1-h+h^2-h^3+h^4)(1+h+h^2+h^3+h^4)(16+8h+4h^2+2h^3+h^4)\\
&\quad \times(81+27h+9h^2+3h^3+h^4)(256+64h+16h^2+4h^3+h^4)(625+125h+25h^2+5h^3+h^4)\\
&\quad \times(2401+343h+49h^2+7h^3+h^4)(4096+512h+64h^2+8h^3+h^4)\\
&\quad \times(6561+729h+81h^2+9h^3+h^4)(556167241916424665616982789+2542408521584455461h^{10}).
\end{aligned}$$

$g(h)$ 仅有8个简单正根，由(13)式及隐函数定理，即得所欲证.

4 高次奇点的周期常数与拟等时中心条件

系统的等时中心首先必须是中心，因此我们按中心条件分三种情形来讨论系统 $(7)_{\delta=0}$ 的复等时中心条件. 在定理2的中心条件中我们记 $a_{32}=b_{32}=r_{32}$.

情形1 中心条件(I)成立.

当 $a_{23}=b_{23}=0$ 时，由文献[11]定理3.1中周期常数的一般递推公式，经过仔细计算得系统 $(7)_{\delta=0}$ 的前20个周期常数为

$$\tau_5=2r_{32},\quad \tau_{10}=\frac{1}{2}(-a_{50}b_{50}+4\lambda),\quad \tau_{15}=0,\quad \tau_{20}=-\frac{\lambda^2}{2}. \tag{14}$$

当 $a_{23}b_{23}\neq 0$ 时，存在复常数 h，使得 $a_{50}=hb_{23}^2$，$b_{50}=ha_{23}^2$，此时系统 $(7)_{\delta=0}$ 的前20个周期常数为

$$\tau_5=2r_{32},\quad \tau_{10}=\frac{1}{2}(-4a_{23}b_{23}-a_{23}^2b_{23}^2h^2+4\lambda),\quad \tau_{15}=\frac{9}{4}a_{23}^2b_{23}^2h,\quad \tau_{20}=4a_{23}^2b_{23}^2. \tag{15}$$

在式(14)和(15)中 $\tau_k=0$，$k\neq 5i$，$i\leq 4$，$i\in N$，且在上述 μ_k 的表达式中已置 $\tau_1=\tau_2=\cdots=\tau_{k-1}=0$，$k=2,3,\cdots,20$.

由式(14)和(15)易见，系统 $(7)_{\delta=0}$ 的原点为复等时中心当且仅当它为普通的线性系统

$$\frac{\mathrm{d}z}{\mathrm{d}T}=z,\quad \frac{\mathrm{d}w}{\mathrm{d}T}=-w.$$

情形2 中心条件(II)成立.

将中心条件(II)代入文献[11]定理3.1中的周期常数递推公式，得到系统 $(7)_{\delta=0}$ 的前40个

周期常数如下

$$
\begin{aligned}
&\tau_5 = 2r_{32}, \quad \tau_{10} = \tfrac{1}{18}(-16a_{41}b_{41} - 9a_{50}b_{50} + 36\lambda), \quad \tau_{15} = 0,\\
&\tau_{20} = \tfrac{1}{2592}(-256a_{41}^2b_{41}^2 + 1152a_{41}a_{50}b_{41}b_{50} - 81a_{50}^2b_{50}^2),\\
&\tau_{25} = -\tfrac{7}{1944}(a_{50}b_{41}^2 + a_{41}^2b_{50})(-32a_{41}b_{41} + 27a_{50}b_{50}),\\
&\tau_{30} = 0, \quad \tau_{35} = \tfrac{56933}{196830}a_{41}^2b_{41}^2(a_{50}b_{41}^2 + a_{41}^2b_{50}),\\
&\tau_{40} = \tfrac{1}{41990400}(-32373248a_{41}^4b_{41}^4 + 4538016a_{41}a_{50}^2b_{41}^5 + 146019648a_{41}^3a_{50}b_{41}^3b_{50}\\
&\qquad - 4238451a_{50}^3b_{41}^4b_{50} + 4538016a_{41}^5b_{41}b_{50}^2 - 4238451a_{41}^4a_{50}b_{50}^3),
\end{aligned}
\tag{16}
$$

其中 $\tau_k = 0,\ k \neq 5i,\ i \leq 8,\ i \in N$. 在上述 μ_k 的表达式中已置 $\tau_1 = \tau_2 = \cdots = \tau_{k-1} = 0,\ k = 2,3,\cdots,40$.

由 $\tau_{35} = 0$ 知存在复常数 h，使得

$$
a_{50} = ha_{41}^2, \quad b_{50} = -hb_{41}^2, \tag{17}
$$

代入 τ_{20} 的表达式得

$$
\tau_{20} = -\frac{1}{2592}a_{41}^2b_{41}^2(256 + 1152a_{41}b_{41}h^2 + 81a_{41}^2b_{41}^2h^4), \tag{18}
$$

由 (18) 式得

$$
h = \frac{4}{3}\sqrt{\frac{-4 \pm \sqrt{15}}{a_{41}b_{41}}}, \tag{19}
$$

将 (17) 式和 (19) 式代入 τ_{40} 的表达式得

$$
\tau_{40} = \frac{1040}{59049}(2396 \mp 619\sqrt{15}\ )a_{41}^4b_{41}^4 \neq 0, \tag{20}
$$

所以当中心条件 (II) 成立时，系统 $(7)_{\delta=0}$ 的原点不是复等时中心.

情形 3　中心条件 (III) 成立.

因为 $a_{41}b_{41} \neq 0$，由中心条件 (III)，可令

$$
a_{23} = rb_{41}, \quad b_{23} = ra_{41}, \quad a_{50} = ha_{41}^2, \quad b_{50} = hb_{41}^2, \tag{21}
$$

其中 r, h 是复常数. 将 (21) 式代入文献 [11] 定理 3.1 中的周期常数递推公式，得到系统 $(7)_{\delta=0}$ 的前 30 个周期常数

$$
\begin{aligned}
&\tau_5 = 2r_{32}, \quad \tau_{10} = \tfrac{1}{2}(-4a_{41}b_{41}r - 4a_{41}b_{41}r^2 - a_{41}^2b_{41}^2h^2 + 4\lambda),\\
&\tau_{15} = \tfrac{1}{4}a_{41}^2b_{41}^2(-1 + 3r)(1 + 3r)h,\\
&\tau_{20} = -\tfrac{1}{96}a_{41}^2b_{41}^2(192r^2 - 192r^3 - 384r^4 - 32a_{41}b_{41}h^2 - 32a_{41}b_{41}rh^2 + 3a_{41}^2b_{41}^2h^4),\\
&\tau_{25} = -\tfrac{1}{1944}a_{41}^3b_{41}^3(11 + 9r)h(-32 + 27a_{41}b_{41}h^2),\\
&\tau_{30} = \tfrac{1}{3110400}a_{41}^3b_{41}^3(2467328 - 6314496r - 2200896a_{41}b_{41}h^2 + 2407104a_{41}b_{41}rh^2\\
&\qquad -607176a_{41}^2b_{41}^2h^4 - 975888a_{41}^2b_{41}^2rh^4 + 73629a_{41}^3b_{41}^3h^6),
\end{aligned}
\tag{22}
$$

其中 $\tau_k = 0,\ k \neq 5i,\ i \leq 6,\ i \in N$. 在上述 μ_k 的表达式中已置 $\tau_1 = \tau_2 = \cdots = \tau_{k-1} = 0,\ k = 2, 3, \cdots, 30$.

由 $\tau_{15} = 0$ 知 $r = \pm\frac{1}{3}$ 或者 $h = 0$，类似于情形2的证明过程容易得到：

当 $r = -\frac{1}{3}$ 时

$$\tau_{25} = -\frac{16}{729}\sqrt{2(4 \pm \sqrt{6})}(8 \pm 3\sqrt{6}\,)\sqrt{a_{41}^5 b_{41}^5} \neq 0;$$

当 $r = \frac{1}{3}$ 时

$$\tau_{25} = -\frac{112}{729}\sqrt{4 \pm \sqrt{15}}(10 \pm 3\sqrt{15}\,)\sqrt{a_{41}^5 b_{41}^5} \neq 0;$$

当 $h = 0$ 时，$\tau_{20} = 2a_{41}^2 b_{41}^2 r^2(1 + r)(-1 + 2r)$，若 $r = \frac{1}{2}$，则

$$\tau_{30} = -\frac{539}{2430} a_{41}^3 b_{41}^3 \neq 0.$$

所以系统 $(7)_{\delta=0}$ 的前30个周期常数为零当且仅当 $\lambda = r_{32} = r = h = 0,\ a_{41}b_{41} \neq 0$ 或者 $\lambda = r_{32} = h = 0,\ r = -1,\ a_{41}b_{41} \neq 0$，即

$$\lambda = a_{32} = b_{32} = a_{23} = b_{23} = a_{50} = b_{50} = 0, \quad a_{41}b_{41} \neq 0, \tag{23}$$

或者

$$\lambda = a_{32} = b_{32} = a_{50} = b_{50} = 0, \quad a_{23} = -b_{41}, \quad b_{23} = -a_{41}, \quad a_{41}b_{41} \neq 0. \tag{24}$$

当条件 (23) 成立时，系统 $(7)_{\delta=0}$ 变为

$$\frac{\mathrm{d}z}{\mathrm{d}T} = \frac{1}{5}z(5 + 3a_{41}z^6w^4 + 2b_{41}z^4w^6), \quad \frac{\mathrm{d}w}{\mathrm{d}T} = -\frac{1}{5}w(5 + 3b_{41}w^6z^4 + 2a_{41}w^4z^6), \tag{25}$$

作变换

$$u = \frac{z(1 + b_{41}z^4w^6)^{\frac{1}{5}}}{(1 + a_{41}z^6w^4)^{\frac{3}{10}}}, \quad v = \frac{w(1 + a_{41}w^4z^6)^{\frac{1}{5}}}{(1 + b_{41}w^6z^4)^{\frac{3}{10}}},$$

则系统 (25) 变为 $\frac{\mathrm{d}u}{\mathrm{d}T} = u,\ \frac{\mathrm{d}v}{\mathrm{d}T} = -v$，故系统 (25) 的原点是复等时中心.

当条件 (24) 成立时，系统 $(7)_{\delta=0}$ 变为

$$\frac{\mathrm{d}z}{\mathrm{d}T} = \frac{1}{5}z(5 + a_{41}z^6w^4 - b_{41}z^4w^6), \quad \frac{\mathrm{d}w}{\mathrm{d}T} = -\frac{1}{5}w(5 + b_{41}w^6z^4 - a_{41}w^4z^6), \tag{26}$$

由极坐标公式 $z = r\mathrm{e}^{\mathrm{i}\theta},\ w = r\mathrm{e}^{-\mathrm{i}\theta}$，沿着系统 (26) 的轨线，$\theta$ 对 T 求导得

$$\frac{\mathrm{d}\theta}{\mathrm{d}T} = \frac{1}{2\mathrm{i}}\left(\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}T} - \frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}T}\right) = -\mathrm{i},$$

即 $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \mathrm{i}\frac{\mathrm{d}\theta}{\mathrm{d}T} = 1$，所以系统 (26) 的原点是复等时中心.

综上所述，我们得以下结论.

定理6 系统 $(7)_{\delta=0}((1)_{\delta=0})$ 的原点 (高次奇点) 是复等时中心 (拟等时中心) 的充分必要条件是条件 (23) 或 (24) 成立.

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Bifurcation of Limit Cycles and Pseudo-isochronous Center at Higher-order Singular Points in a Septic Polynomial System

WU Yu-sen, LI Pei-luan

(School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471003)

Abstract: Investigated in this paper are the center conditions, pseudo-isochronous center conditions and bifurcation of limit cycles at higher-order singular points for a class of septic system. Firstly, the higher-order singular point is transformed into the origin by a homeomorphic transformation and a complex transformation. Then the first 45 singular point quantities at the origin are calculated and the conditions for the higher-order singular point to be a center and the highest order fine focus are deduced as well. With these conclusions, a septic system which allows the appearance of 8 limit cycles in the neighborhood of higher-order singular points is constructed. Finally, a new algorithm is applied to find necessary conditions for pseudo-isochronous centers, and then the sufficiency of these conditions is proved.

Keywords: higher-order singular point; bifurcation of limit cycles; pseudo-isochronous center; septic system

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